On the Solution of Linear Equations in Infinitely Many Variables by Successive Approximations.*

By J. L. Walsh.

In this paper we shall consider systems of equations of the type

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1 , \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots = c_2 , \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots = c_3 , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot , \end{array}\right\} \tag{1}$$

where the a_{ij} and c_i are given real or complex numbers and the x_i are to be determined. Systems of type (1) have been solved by various means,† including the method of successive approximations, but this method has been used chiefly for Hilbert space [i. e., the space of points $\{x_k\}$ for which $\sum\limits_{k=1}^{\infty} | x_k |^2$ converges] with corresponding restrictions on the a_{ij} and c_i.‡ It is the object of the present paper to give a number of new conditions under which (1) can be solved by successive approximations; in particular, it is shown that if (1) has a non-vanishing normal determinant and if a simple transformation of the system is made, then the method of successive approximations can be used. The method of successive approximations is very convenient for numerical computation.

We shall use the method of successive approximations to prove the following theorem, which applies to a system of equations of type slightly less general than (1):

Theorem I. *If there exist positive constants C, M, and P such that the coefficients of the system*

$$\left.\begin{array}{r} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1 , \\ x_2 + a_{23}x_3 + \cdots = c_2 , \\ x_3 + \cdots = c_3 , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot , \end{array}\right\} \tag{2}$$

satisfy the restrictions

* Presented to the American Mathematical Society (Chicago), Apr. 7, 1917.

† See, e. g., F. Riesz, *Équations Linéaires.*

‡ See E. Goldschmidt, Würzburg Dissertation (1912).

$|c_k| \leqq MC^k$, $\sum_{j=k+1}^{\infty} |a_{kj}|$ *convergent* $(k=1, 2, \cdots)$, $\sum_{j=k+1}^{\infty} |a_{kj}| \leqq P$ *for every* $k > K$, $C < (1/P)$, $C \leqq 1$, *then* (2) *has one solution and only one solution for which* $|x_k| \leqq \mu\gamma^k$, $\gamma < (1/P)$, $\gamma \leqq 1$.

We first consider the special case $K=0$, and we take for approximations

$$\left.\begin{aligned} &x_k^{(1)} = c_k \quad (k=1, 2, \cdots), \\ &x_k^{(i+1)} = c_k - [a_{kk+1}x_{k+1}^{(i)} + a_{kk+2}x_{k+2}^{(i)} + \cdots] \quad (i=1, 2, \cdots). \end{aligned}\right\} \tag{3}$$

From (3) it follows that

$$\left.\begin{aligned} x_k^{(1)} &= c_k, \\ x_k^{(2)} - x_k^{(1)} &= -[a_{k,\,k+1}x_{k+1}^{(1)} + a_{k,\,k+2}x_{k+2}^{(1)} + \cdots], \\ x_k^{(3)} - x_k^{(2)} &= -[a_{k,\,k+1}(x_{k+1}^{(2)} - x_{k+1}^{(1)}) + a_{k,\,k+2}(x_{k+2}^{(2)} - x_{k+2}^{(1)}) + \cdots], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\} \tag{4}$$

and therefore

$$x_k = x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots \tag{5}$$

$$<< MC^k + PMC^{k+1} + P^2MC^{k+2} + \cdots = MC^k/(1-PC). \tag{6}$$

The x_k as defined by (5) are a solution of (2), for if we add all the equations of (4) and sum by columns the resulting absolutely convergent double series, we have

$$x_k = c_k - [a_{kk+1}x_{k+1} + a_{kk+2}x_{k+2} + \cdots].$$

By (6) we see that for the x_k defined by (5) there exist μ and γ such that $|x_k| \leqq \mu\gamma^k$, $\gamma < (1/P)$, $\gamma \leqq 1$. Under this restriction the solution is unique, for if x_k' and x_k'' denote two solutions satisfying this restriction, their difference $x_k = x_k' - x_k''$ is a solution of the homogeneous system corresponding to (2):

$$\left.\begin{aligned} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots &= 0, \\ x_2 + a_{23}x_3 + \cdots &= 0, \\ x_3 + \cdots &= 0, \\ \cdots\cdots &, \end{aligned}\right\} \tag{7}$$

and we have

$$|x_k| \leqq NX^k, \quad X < (1/P), \quad X \leqq 1.$$

Place

$$\begin{aligned} &x_k^{(1)} = x_k \quad (k=1, 2, \cdots), \\ &x_k^{(i+1)} = -[a_{kk+1}x_{k+1}^{(i)} + a_{kk+2}x_{k+2}^{(i)} + \cdots] \quad (i=1, 2, \cdots). \end{aligned}$$

from which it follows that $|x_k^{(i+1)}| \leqq P^i N X^{k+i}$. Hence

$$\lim_{i=\infty} x_k^{(i)} = 0.$$

But we have $x_k^{(i)} = x_k^{(1)} = x_k$ by equations (7). Hence $x_k = 0$, which proves the uniqueness of the solution and completes the proof of Theorem I for the case $K = 0$. The reader will easily complete the proof of Theorem I in its generality $[K \neq 0]$ by the use of mathematical induction. In this proof, it will appear that equations (3) and (5) will give x_k for every value of k.

When the x_k defined by (3) and (5) are computed in terms of the a_{ij} and c_i, it is found that

$$x_k = c_k - \sum_{j=k+1}^{\infty} a_{kj} c_j + \sum_{j=k+1}^{\infty} \sum_{i=j+1}^{\infty} a_{kj} a_{ji} c_i - \cdots, \tag{8}$$

which is the so-called Neumann series.

The following special case of Theorem I will be used in the sequel:

Theorem II. *If for the system* (2) *we have* $\sum_{j=k+1}^{\infty} |a_{kj}|$ *convergent for every* k, $\sum_{j=k+1}^{\infty} |a_{kj}| \leqq P < 1$ *for every* $k > K$,

$$|c_k| \leqq C \text{ for every } k,$$

then (2) *has one solution and only one solution for which the* x_k *are bounded. Moreover, this solution is given by the Neumann series* (8).*

We now return to the system of general type (1), and shall proceed to show that if (1) has a non-vanishing normal determinant and if the c_k are bounded, then (1) can be transformed into an equivalent system of type (2). This latter system will be shown to satisfy the hypotheses of Theorem II, and therefore the method of successive approximations can be used. To show the possibility of making this transformation we need the

Lemma. *In any non-vanishing determinant*

$$\Delta^{(k)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix},$$

* Theorem II is similar to a theorem given by von Koch, *Jahresbericht*, 1913, p. 289.

the rows can be arranged so that no minor

$$\Delta^{(i)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix} \quad (i = 1, 2, \cdots k)$$

will vanish.

This lemma is evidently true (although trivial) for $k = 1$ and $k = 2$; the reader can readily complete the proof by induction.

We have supposed (1) to have a non-vanishing normal determinant; that is, we suppose $\sum_{i,j=1}^{\infty} | a_{ij} - d_{ij} |$ to be convergent [d_{ij} is the Kronecker symbol whose value is zero or unity according as $i \neq j$ or $i = j$] and

$$\lim_{n=\infty} \Delta^{(n)} = \Delta \neq 0.$$

Then there exists k such that $\Delta^{(n)} \neq 0$ for $n = k, k + 1, \cdots$. Hence, by the Lemma, the order of the equations can be changed (if necessary) so that $\Delta^{(n)} \neq 0$ for $n = 1, 2, \cdots$. Such rearrangement will not affect the convergence of the double series $\sum_{i,j=1}^{\infty} | a_{ij} - d_{ij} |$ nor the value of Δ.

We suppose, now, that this arrangement has been made, and we proceed to transform (1) into an equivalent system of the type

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + \cdots &= \beta_1 , \\ b_{22}x_2 + b_{23}x_3 + \cdots &= \beta_2 , \\ b_{33}x_3 + \cdots &= \beta_3 , \\ \cdots\cdots\cdots \end{aligned}\right\} \tag{9}$$

This transformation is made by placing* $b_{1k} = a_{1k}$, $\beta_1 = c_1$, and for $n > 1$,

$$b_{n,k} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{n,k} \end{vmatrix}, \quad (k = 1, 2, \cdots)$$

$$\beta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & c_n \end{vmatrix}.$$

The β_n as thus defined are bounded for if we set

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* Riesz, *l. c.*, p. 11.

$$A_k = \sum_{i=1}^{\infty} | a_{ik} - d_{ik} |, \quad \Pi = \prod_{k=1}^{\infty} (1 + A_k),$$

we shall have

$$| \beta_n | = \text{absolute value of} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,\,n-1} & c_1 \\ . & . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{n,\,n-1} & c_n \end{vmatrix} \leqq \Pi \cdot C, \tag{10}$$

where $| c_k | \leqq C$ $(k = 1, 2, \cdots)$.

Similarly, we shall prove that $\sum_{j=i+1}^{\infty} | b_{ij} |$ converges for every i. For by using a method similar to that used in (10), it follows that for $n < k$,

$$| b_{nk} | \leqq \Pi \, [\, | a_{1k} | + | a_{2k} | + \cdots + | a_{nk} | \,].$$

Hence we have

$$\begin{aligned} & | b_{n,\,n+1} | + | b_{n,\,n+2} | + \cdots \\ & \leqq \Pi \, [\, | a_{1,\,n+1} | + | a_{1,\,n+2} | + \cdots \\ & \qquad + | a_{2,\,n+1} | + | a_{2,\,n+2} | + \cdots \\ & \qquad + \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & \qquad + | a_{n,\,n+1} | + | a_{n,\,n+2} | + \cdots] \, . \end{aligned}$$

Then for every value of n, the series of absolute values of the coefficients b_{nk}, $k > n$, converges; and its sum approaches zero as n becomes infinite.

None of the b_{nn} is zero and they approach a limit different from zero, so when we write system (9) in the form (2) the hypothesis of Theorem II is satisfied. Therefore we have shown that if (1) has a non-vanishing normal determinant and the c_i are bounded, and if (1) is transformed into an equivalent system of type (2), then the method of successive approximations can be used. Incidentally we have proved the theorem of von Koch that if (1) has a non-vanishing normal determinant and if the c_i are bounded, then (1) has one solution and only one solution such that the x_i are bounded.

We shall now state two theorems, each of which is proved precisely as Theorem I was proved. The x_k of each theorem are given by the Neumann series (8).

Theorem III. *If system* (2) *is such that* $\sum_{j=k+1}^{\infty} | a_{kj} |^p$ $(p > 1)$ *converges for every* k, $\sum_{j=k+1}^{\infty} | a_{kj} |^p \leqq Q^p$ *for every* $k > K$,

$$| c_k | \leqq M C^k, \qquad C < \frac{1}{(1 + Q^{p/(p-1)})^{(p-1)/p}}$$

then (2) *has one solution and only one solution for which*

$$| x_k | \leqq \mu\gamma^k, \qquad \gamma < \frac{1}{(1 + Q^{p/(p-1)})^{(p-1)/p}}. \text{ *}$$

Theorem IV. *Under the restrictions* $| a_{ik} | \leqq NT^{k-i}$ *for every* $k > i$,

$$| c_k | \leqq MC^k, \quad TC < 1/(1+N),$$

system (2) *has one solution and only one solution for which*

$$| x_k | \leqq \mu\gamma^k, \quad T\gamma < 1/(1+N). \dagger$$

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* A slightly less general theorem for the case $p = 2$ was proved by von Koch using infinite determinants. See *Proc. Camb. Cong. Math.* (1912) I, p. 354.

In proving Theorem III there will be found useful the following inequality due to Hölder:

$$\left| \sum_{k=1}^{\infty} a_k b_k \right|^p \leqq \left(\sum_{k=1}^{\infty} | a_k |^p\right) \left(\sum_{k=1}^{\infty} | b_k |^{\frac{p}{p-1}}\right)^{p-1}.$$

See Riesz, *l. c.*, p. 45.

† Cf. von Koch, *l. c.*, p. 355.